ON A SOLUTION OF THE PROBLEM OF AN UNDERGROUND EXPLOSION IN SOFT SOILS

(K RESHENIIU ZADACHI O PODZEMNOM VZRYVE V MIAGKIKH GRUNTAKH)

PMM Vol.28, № 6, 1964, pp. 1070-1082

S.S. GRIGORIAN
(Moscow)

(Received June 16, 1964)

The solution of the problem of explosion of high explosive charges in the ground is of interest for the needs of seismic prospecting, mining, or production of underground cavities, by an explosion method. In various cases it is necessary to have information of the different characteristics of this solution, on the energy and mode of the elastic waves emitted by the seat of origin of the explosion; on the dynamics of the formation of the explosion cavity, on the stress and velocity fields in the zones near the explosion where inelastic behavior of the ground is manifested essentially, etc. Some approximate solutions of this problem are known [1 to 7], which are based on very strong simplifying schematizations of the phenomena and which do not give sufficiently satisfactory quantitative and qualitative results. An attempt is made below to give a more rigorous formulation and effective solution of the problem of an underground explosion by using existing theoretical conceptions of the behavior of ground [8 to 10], as well as the results of experiments formulated especially on their basis [11 to 15].

1. The equations of the problem for centrally symmetric motion of the ground are, as follows from [9]:

$$\frac{\partial\theta}{\partial t} + v\frac{\partial\theta}{\partial r} + (1-\theta)\left(\frac{\partial v}{\partial r} + \frac{2v}{r}\right) = 0$$

$$\rho_0\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) - (1-\theta)\left[\frac{\partial\sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\varphi)\right] = 0$$

$$\frac{\partial\theta_*}{\partial t} + v\frac{\partial\theta_*}{\partial r} = \left(\frac{\partial\theta}{\partial t} + v\frac{\partial\theta}{\partial r}\right)e\left(\frac{\partial\theta}{\partial t} + v\frac{\partial\theta}{\partial r}\right)e(\theta-\theta_*) \qquad (1.1)$$

$$2G\left[\frac{\partial v}{\partial r} - \frac{1}{3}\left(\frac{\partial v}{\partial r} + \frac{2v}{r}\right)\right] = \frac{\partial(\sigma_r + p)}{\partial t} + v\frac{\partial(\sigma_r + p)}{\partial r} + \Lambda(\sigma_r + p)$$

$$G\left[\frac{v}{r} - \frac{1}{3}\left(\frac{\partial v}{\partial r} + \frac{2v}{r}\right)\right] = \frac{\partial(\sigma_\varphi + p)}{\partial t} + v\frac{\partial(\sigma_\varphi + p)}{\partial r} + \Lambda(\sigma_\varphi + p)$$

$$2F(p)\Lambda = \left[\frac{4}{3}G(\sigma_r - \sigma_\varphi)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) - F'(p)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial r}\right)\right]e\,[J_2 - F(p)]\,e\left[\frac{4}{3}G(\sigma_r - \sigma_\varphi)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) - F'(p)\left(\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial r}\right)\right]$$

$$e(z) = \begin{cases} 1 & (z \geqslant 0) \\ 0 & (z < 0) \end{cases}, \quad 2J_2 \equiv (\sigma_r + p)^2 + 2(\sigma_\varphi + p)^2 \qquad \text{(1.1) cont.}$$

$$-3p \equiv \sigma_r + 2\sigma_\varphi, \qquad p - p_0 = f(\theta, \theta_*) \qquad \left(\theta \equiv 1 - \frac{\rho_0}{\rho}, \ \theta_* \equiv 1 - \frac{\rho_0}{\rho_*}\right)$$

The initial and boundary conditions of the problem are

$$\sigma_r(r, 0) = \sigma_\varphi(r, 0) = -p_0, \qquad v(r, 0) = 0$$
$$\theta(r, 0) = \theta_*(r, 0) = 0, \qquad r \geqslant a_0 \tag{1.2}$$

$$\sigma_r(a, t) = -\psi(a), \qquad v(a, t) = da/dt$$
$$\sigma_r(r, t) \to -p_0, \qquad \sigma_\varphi(r, t) \to -p_0, \qquad v(r, t) \to 0 \tag{1.3}$$
$$\theta(r, t) \to 0, \qquad \theta_*(r, t) \to 0 \quad \text{for } r \to \infty$$

Here r and t are the coordinate and time; v is velocity; σ_r and σ_φ are the principal stresses; ρ and ρ_0 are the present and initial densities; G is the shear modulus, and p_0 is the initial pressure in the ground; the functions $F(p)$ and $f(\theta, \theta_*)$ describe the condition of the plasticity and volume compressibility of the ground.

The function $\psi(a)$ in the first of conditions (1.3) expresses the customarily used hypothesis on adiabatic and quasi-stationary change in the state of the gaseous explosion products and, for the customary high explosive (as TNT, say), may be taken in the form

$$\psi(a) = p_{00}\left(\frac{a}{a_0}\right)^{-3\gamma} \quad (\text{for TNT } \gamma \approx 1.25, \ p_{00} \approx 2 \times 10^4 \text{ kg/cm}^2) \tag{1.4}$$

The unknown function $a = a(t)$ describes the law of boundary motion of the explosion cavity and should be found during the process of solving the problem.

As the experiments [11 to 15] show, the function $F(p)$ for sand, loam, loess, clay (and only such ground will be considered herein) is

$$6F(p) = (kp + b)^2 \tag{1.5}$$

in a range of variation of p from small values to hundreds of atmospheres. The pressure p in the explosion problem varies between several tens of thousands of atmospheres (the value of p_{00} in (1.4)) to small values on the order of p_0.

For large values of p (apparently on the order of thousands and tens of thousands of atmospheres), the function $F(p)$ should remain bounded [10 and 16], i.e. the relationship (1.5) cannot be used throughout in solving the problem of an explosion in ground set in motion at all times. There are not yet specific experimental results on the value of p, starting with which $F(p)$ ceases to increase. However, taking into account the abrupt drop in all the stresses, particularly in p in a small neighborhood of the explosion cavity in the first instants after the detonation of the high-explosive charge, it can apparently be assumed without great error, that the relationship (1.5) holds for all values of p encountered in the problem.

In the first instants, when a small region of ground around the explosion cavity is in motion this region will, as may be shown, be separated from ground not yet involved in the motion by a spherical surface of strong discontinuity, a shock wave being propagated into the ground at rest. The state of the ground particles directly behind this surface is determined by the dependence of σ_r on $\theta = \theta_*$, which has been obtained in the analysis of uniaxial deformation of a ground element, and which, for large values of σ_r, may correspond to strain with elastic shear, i.e. we will have $J_2 < F(p)$; for smaller values of σ_r, may correspond to strain with plastic shear, i.e. $J_2 = F(p)$. The value of σ_r delimiting these two states is determined [16 and 17] by the form of the function $F(p)$ and $p_* - p_0 = \chi(\theta_*) \equiv f(\theta_*, \theta_*)$.

After the shock wave has passed the particle, this latter starts to be moved toward increasing r, shear strains develop, and at a certain time they become such that plastic flow sets in, i.e. for large values of σ_r the elastic strains for which $\Lambda = 0$ will hold only in a certain layer adjacent to the shock-wave surface. Because of the large value of $G (G \sim 10^3 \text{ kg/cm}^2)$, as is seen from (1.1), the elastic shear strains may only hold for small values of particle displacement and it can be expected that this layer will be relatively thin. Hence, to simplify the mathematical problem, the existence of this layer can be neglected and we can consider that the plastic flow starts directly behind the front of the shock wave. This assumption probably introduces a negligibly small error into the solution of the problem.

In the expression for Λ (see (1.1)), the difference is $\sigma_r - \sigma_\varphi < 0$ for $J_2 = F(p)$. As will be seen from the solution to the problem constructed below, the inequalities $\partial p / \partial t + v \partial p / \partial r < 0$, $\partial v / \partial r < 0$ also hold in the initial stages of the motion. Moreover, $v > 0$ in the initial stage. All this means that if the plasticity condition is satisfied on the shock-wave surface, it is also retained everywhere behind this surface in the initial stage.

The assumptions made permit simplification of the original system of Equations (1.1).

Henceforth, we shall use a simplified system, as

$$\frac{\partial\theta}{\partial t} + v\frac{\partial\theta}{\partial r} + (1-\theta)\left(\frac{\partial v}{\partial r} + \frac{2v}{r}\right) = 0, \qquad \frac{\partial\theta_*}{\partial t} + v\frac{\partial\theta_*}{\partial r} = 0$$

$$\rho_0\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) - (1-\theta)\left[\frac{\partial\sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\varphi)\right] = 0 \qquad (1.6)$$

$$\sigma_\varphi = \alpha\sigma_r + \beta, \quad -3p = \sigma_r + 2\sigma_\varphi, \quad \alpha = \frac{3\sqrt{2} - k}{3\sqrt{2} + 2k}, \quad \beta = \frac{3b}{3\sqrt{2} + 2k}$$

$$\theta = \theta_* + \varepsilon\varphi(p - p_0, \theta_*),$$

Here ε is a small parameter $(\varepsilon \ll 1)$. The smallness of the parameter ε is associated with the fact that, as experiments show, the discharging branches of the $\theta = \theta(p - p_0, \theta_*)$ diagram are very insignificantly inclined to the p-axis for the ground being considered.

Hence, the characteristic value of this slope may be introduced as ε and it may be considered that $\varphi(p - p_0, \theta_*) \sim 1$.

Conditions (1.2) and (1.3) change partially: conditions of the shock wave

$$-\sigma_{r*} \equiv -\sigma_r(R, t) = p_0 + \rho_0\theta_*(dR/dt)^2, \quad v_* \equiv v(R, t) = \theta_* dR/dt \qquad (1.7)$$

replace the conditions for $r \to \infty$.

Here $r = R(t)$ is the unknown law of shock-wave motion. The shock wave will attenuate as time passes, its propagation velocity dR/dt will drop and a time will arrive when it will become equal to the propagation velocity of small elastic disturbances in the undeformed ground at rest ahead of the shock wave

$$c_0 = [-\rho_0^{-1} d\sigma_r(\theta) / d\theta]_{\theta=0}^{1/2}$$

Here $\sigma_r = \sigma_r(\theta)$ is a dependence corresponding to uniaxial strain. In the notation customary in the elasticity theory, we will have for c_0

$$\rho_0 c_0^2 = \lambda + 2\mu = \lambda + 2G = K + \tfrac{4}{3}G, \qquad K = (dp / d\theta)_{\theta=0} \qquad (1.8)$$

Here $\lambda, \mu = G$ are Lamé coefficients.

Starting with this time, a slight motion starts ahead of the shock wave, emission of an elastic wave by the shock wave-front begins and the subsequent construction of the problem should take this wave into account. Experiments show [11 to 13] that small elastic disturbances actually emerge to the front at some distance from the seat of the explosion during propagation of the explosion wave.

The possibility of the emission of an elastic wave is determined by the properties of the dependence $\sigma_x = \sigma_x(\theta)$ obtained from the initial undeformed state during uniaxial strain. For small strains, this will be the elastic relationship

$$-\sigma_{xe}(\theta) = p(\theta) + {}^4/_3 G\theta, \qquad \theta = -\frac{\partial w}{\partial x} \quad (u \text{ is the displacement}) \qquad (1.9)$$

If plastic shear strain starts with increasing θ, the relationship (1.9) is replaced by another, obtained by using the plasticity condition (1.5)

$$-\sigma_{xp}(\theta) = mp(\theta) + q, \qquad m = \frac{3}{1+2\alpha}, \qquad q = \frac{2\beta}{1+2\alpha} \qquad (1.10)$$

Evidently the transition to plastic shear is possible when the curves (1.9) and (1.10) intersect in the $\sigma_x \theta$ plane at the point ($\theta = \theta_1$, $\sigma_x = -\sigma_1$).

If the condition

$$K < \frac{2\sqrt{2}G}{k} \qquad (1.11)$$

is satisfied, such an intersection always occurs. If inequality (1.11) is not satisfied, there will be no intersection at $p''(\theta) \geqslant 0$ for any value of θ; it may occur in this case if $p''(\theta)$ is negative for small values of θ and positive for large values of θ. In all cases when there is an intersection, the uniaxial strain diagram will agree with (1.9) for $\theta < \theta_1$ and with (1.10) for $\theta > \theta_1$; however, plastic shear may again set in [16 and 17] for significant p.

The quantity $-d\sigma_x/d\theta$ will decrease by a jump at the point of intersection. If $p(\theta) = K\theta$ (linear elasticity) for $\theta \leqslant \theta_1$, the perturbations ahead of the shock will be described by linear equations of the theory of elasticity, if the dependence $p(\theta)$ is nonlinear, then the equations describing the perturbations ahead of the shock will be nonlinear also. However, this nonlinearity is usually negligible and it is apparently possible to limit oneself to the linear case.

The solutions of the problem should be constructed by joining the solution of (1.6) to the solution of the equations describing the perturbations ahead of the shock by means of the conditions on the shock, which will differ from (1.7) because of the perturbations ahead of the wave. However, these conditions will be inadequate because the perturbations ahead of the shock are themselves unknown, hence, still another (precisely one, as will be seen from the subsequent exposition) condition should be appended.

The choice of this condition for the one-dimensional plane motion case is elementary [18 and 17] and reduces to the following. If the inequality $p''(\theta) > 0$ is satisfied for $0 \leqslant \theta < \theta_1$, then $\sigma_x = -\sigma_1$ will be this condition. If the inequality $p''(\theta) < 0$ is satisfied for $0 \leq \theta < \theta_1$ then this condition reduces to the equality

$$\frac{dR}{dt} - v_e = c_e \equiv \left[-\frac{1}{\rho_0} \frac{d\sigma_{xe}(\theta)}{d\theta} \right]^{1/2}$$

for $0 \leqslant \theta < \theta_1$ and to the equality $\sigma_x = -\sigma_1$ for $\theta = \theta_1$; in the latter case the inequality

$$dR / dt - v_e < c_e(\theta_1)$$

will be satisfied.

Here v_e is the particle velocity directly in front of the wave. It is much less than dR/dt and c_e, hence it may be omitted from the inequalities.

The situation is more complicated in the case under consideration because a $\sigma_r = \sigma_r(\theta)$ does not exist here (σ_r depends on two arguments $\partial u/\partial r$ and u/r; u is the displacement), whose properties determine the selection of the additional condition. However, considerations on shock-wave stability and evolutionarity of the solution, which permit making the selection of this condition in the plane case [17], are easily realized in this case too. If $p''(\theta) < 0$ for $0 \leqslant \theta < \theta_1$, the stability condition again leads to the relation

$$\frac{dR}{dt} = \left[-\frac{1}{\rho_0} \frac{\partial \sigma_{re}(\varepsilon_r, \varepsilon_\varphi)}{\partial \varepsilon_r} \right]^{1/2}, \qquad \varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\varphi = \frac{u}{r} \tag{1.12}$$

Since the perturbation ahead of the shock is propagated into a medium at rest, it will be a one-direction wave (divergent), hence conditions (1.12) are sufficient for the solution of the problem in the case considered. If $p''(\theta) \geq 0$ for $0 \leqslant \theta < \theta_1$, the leading edge of the perturbation will also be a shock wave. The intensity of this wave, as well as the continuous motion behind it up to the fundamental shock, are determined by a function which is unknown and is determined, if one missing condition connecting the motion parameters directly ahead of the fundamental shock, is appended. It can be shown that the plasticity condition (1.5) will be this condition.

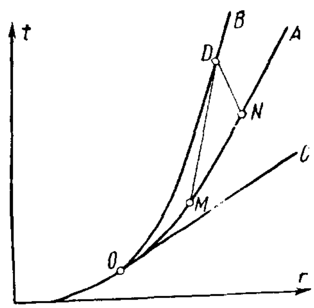

Fig. 1

Let us assume that this is not so, i.e. to the right of the line $r = R(t)$ in the rt plane, there is a left boundary line of the domain of motion where the shear proceeds elastically (line OA, Fig.1). Condition (1.5) is satisfied in the region between this line and the line $r = R(t)$; the shear proceeds plastically. The case when the shear is elastic everywhere to the right and up to the line $r = R(t)$ is evidently impossible since $dR/dt > c_e$ would hold for $p''(\theta) \geqslant 0$ $(0 \leqslant \theta < \theta_1)$, which is incompatible with the fact that the line $r = R(t)$ must pass through the point O (time of emission of the elastic wave) into the domain where $dR/dt < c_{*0}$ and $c_* \geqslant c_{*0}$ (c_{*0} is the value of c_* directly behind the leading edge of the perturbation, the slope of the line OC). By virtue of condition (1.11), the propagation velocity of the small perturbations in the plastic domain BOA will be smaller than in the elastic domain AOC. It is also evident that the propagation velocity of the boundary state at each point on OA (the slope of the line OA) should be included between the values of these perturbation velocities at the points on OA .

Let us now consider the characteristics of the two families MD and ND for the plastic equations. Evidently, if the solution on MN is known, we have a Cauchy problem in the triangle MDN , which is solved uniquely and determines the values of the desired functions at the point D on the shock wave OB , which makes the conditions at its point D completely defined and sufficient for the construction of the solution to the left of the shock. Moreover, if the line OA is prescribed, the solution is completely defined in the elastic domain AOC also. Hence, assignment of this line completely defines the solution of the problem as a whole, where such assignment may be completely arbitrary, and the solution is not unique. The solution of the problem turns out not to be evolutionary, i.e. assignment of the desired functions at a certain time $t = t_1$ does not determine the solution uniquely in subsequent times. The contradiction is eliminated if it is assumed that the plastic domain BOA does not exist, i.e. that condition (1.5) is satisfied only directly ahead of the shock wave on the line OB . This means that relationship (1.5) should be taken as the missing condition connecting the values of the motion parameters on the shock.

For simplicity, we shall henceforth consider linear relationships to describe the motion ahead of the shock. In this case, the dependences between the stresses and velocities of the deformations which are obtained from (1.1) for $\Lambda \equiv 0$ and $\theta_* = \theta \ll 1$ transform into Hooke's law after integration with respect to the time

$$\sigma_{re} = \left(K - \frac{2}{3}G\right)\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) + 2G\frac{\partial u}{\partial r} - p_0$$

$$\sigma_{\varphi e} = \left(K - \frac{2}{3}G\right)\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) + 2G\frac{u}{r} - p_0 \tag{1.13}$$

As is known, the solution of the problem of a diverging centrally symmetric elastic wave is expressed in terms of one function by means of Formula

$$u_e = \frac{\partial}{\partial r}\frac{\Phi(c_0 t - r)}{r} = -\frac{\Phi}{r^2} - \frac{\Phi'}{r}, \qquad v_e \equiv \frac{\partial u}{\partial t} = -c_0\left(\frac{\Phi''}{r} + \frac{\Phi'}{r^2}\right)$$

$$\sigma_{re} = \rho_0 c_0^2\left[\frac{1}{r}\Phi'' + 4\gamma^2\left(\frac{1}{r^2}\Phi' + \frac{1}{r^3}\Phi\right)\right] - p_0$$

$$\sigma_{\varphi e} = \rho_0 c_0^2\left[\frac{1-2\gamma^2}{r}\Phi'' - 2\gamma^2\left(\frac{1}{r^2}\Phi' + \frac{1}{r^3}\Phi\right)\right] - p_0$$

$$\gamma^2 = \frac{G}{K + {}^4/_3 G} = \frac{1-2\sigma}{2(1-\sigma)} \tag{1.14}$$

Here γ is the ratio of the transverse and longitudinal elastic wave velocities; σ is Poisson's ratio, the primes on the Φ denote differentiation. The conditions on the shock $r = R(t)$ are written as

$$-(\sigma_{r*} - \sigma_{re*}) = \rho_0\frac{\theta_* - \theta_{e*}}{(1-\theta_{e*})^2}(R^{\cdot} - v_{e*})^2, \qquad \sigma_{\varphi*} = \alpha\sigma_{re*} + \beta$$

$$R^{\cdot}(\theta_{e*} - \theta_*) = (1-\theta_*)v_{e*} - (1-\theta_{e*})v_* \tag{1.15}$$

Here and henceforth the dot will denote differentiation with respect to time and the values of the quantities on the shock will bear the subscript * Since $\theta_e \ll 1$, in the elastic wave, conditions (1.15) may be simplified

$$-(\sigma_{r*}-\sigma_{re*})=\rho_0(\theta_*-\theta_{e*})(R^{\cdot}-v_{e*})^2, \qquad \sigma_{\varphi e*}=\alpha\sigma_{re*}+\beta$$
$$R^{\cdot}(\theta_*-\theta_{e*})=v_*-(1-\theta_*)v_{e*} \tag{1.16}$$

It is easy to show that the expression Λ for particles directly in front of the shock will be zero. In the elastic wave itself, i.e. for $r > R(t)$, the condition that the shear proceeds elastically $J_2 < F(p)$, reduces to

$$\left|\frac{1}{r}\Phi''(\xi)+\frac{3}{r^2}\Phi'(\xi)+\frac{3}{r^3}\Phi(\xi)\right|<\left|\frac{kK}{2\sqrt{2}\,Gr}\Phi''(\xi)-\frac{b-p_0}{2\sqrt{2}\,G}\right| \tag{1.17}$$

This inequality turns into an equality at the point of each characteristic $\xi = a_0 t - r$ = const for $r = R$. Since the left-hand side in (1.17) approaches zero for ξ = const and $R < r \to \infty$, and the right-hand side approaches a positive quantity it may be expected that condition (1.17) will be satisfied, i.e. that plastic shear domains will not occur ahead of the shock (this latter circumastance would substantially complicate the solution of the problem). Of course, compliance with condition (1.17) everywhere in the domain $r > R(t)$ should be confirmed in the actual construction of the solution.

As time passes, a time t_1 sets in when the shock wave is completely exhausted, i.e. the equalities $v_* = v_{e*}$, $\theta_* = \theta_{e*}$, $\sigma_{r*} = \sigma_{re*}$, will be satisfied at $r = R_\infty = R(t_1)$, after which the motion will be continuous everywhere, hence it will be a diverging elastic wave of (1.14) type in the domain $r > R_\infty$, which should join continuously with the solution of (1.1) in the domain $r < R_\infty$.

2. Let us seek the solution of Equation (1.6) as power series in the small parameter ε, i.e.

$$\begin{gathered} v = v^0+\varepsilon v^1+\ldots,\ p=p^0+\varepsilon p^1+\ldots,\ \theta_*=\theta_*{}^0+\varepsilon\theta_*{}^1 \\ \sigma_r=\sigma_r{}^0+\varepsilon\sigma_r{}^1+\ldots,\ \sigma_\varphi=\sigma_\varphi{}^0+\varepsilon\sigma_\varphi{}^1\ldots,\ \theta=\theta^0+\varepsilon\theta^1+\ldots \end{gathered} \tag{2.1}$$

Substituting (2.1) into (1.6), we obtain first-approximation equations

$$\frac{\partial\theta^0}{\partial t}+v^0\frac{\partial\theta^0}{\partial r}+(1-\theta^0)\left(\frac{\partial v^0}{\partial r}+\frac{2v^0}{r}\right)=0, \qquad \frac{\partial\theta_*{}^0}{\partial t}+v^0\frac{\partial\theta_*{}^0}{\partial r}=0$$
$$\rho_0\left(\frac{\partial v^0}{\partial t}+v^0\frac{\partial v^0}{\partial r}\right)-(1-\theta^0)\left[\frac{\partial\sigma_r{}^0}{\partial r}+\frac{2}{r}(\sigma_r{}^0-\sigma_\varphi{}^0)\right]=0$$
$$\sigma_\varphi{}^0=\alpha\sigma_r{}^0+\beta, \qquad -3p^0=\sigma_r{}^0+2\sigma_\varphi{}^0, \qquad \theta^0=\theta_*{}^0 \tag{2.2}$$

The second approximation equations are

$$\frac{\partial\theta^1}{\partial t}+v^0\frac{\partial\theta^1}{\partial r}+v^1\frac{\partial\theta^0}{\partial r}+(1-\theta^0)\left(\frac{\partial v^1}{\partial r}+\frac{2v^1}{r}\right)=0$$
$$\frac{\partial\theta_*{}^1}{\partial t}+v^0\frac{\partial\theta_*{}^1}{\partial r}+v^1\frac{\partial\theta_*{}^0}{\partial r}=0 \tag{2.3}$$

$$\rho_0\left(\frac{\partial v^1}{\partial t}+v^0\frac{\partial v^1}{\partial r}+v^1\frac{\partial v^0}{\partial r}\right)-(1-\theta^0)\left[\frac{\partial\sigma_r^1}{\partial r}+\frac{2}{r}(\sigma_r^1-\sigma_\varphi^1)\right]+ \qquad \text{(2.3) cont.}$$

$$+\theta^1\left[\frac{\partial\sigma_r^0}{\partial r}+\frac{2}{r}(\sigma_r^0-\sigma_\varphi^0)\right]=0, \qquad \sigma_\varphi^1=\alpha\sigma_r^1$$

$$p^1=-\frac{1+2\alpha}{3}\sigma_r^1, \qquad \theta^1=\theta_*^1-\varphi(p^0-p_0,\theta_*^0)$$

We do not write down the equations for the subsequent approximations, they are awkward and not needed, in practice, for the solution of the problem. The system (2.2) is nonlinear and agrees with the equations of the problem for the particular case of a medium with the property $d\theta/dt=0$ for $dp/dt<0$ (an incompressible medium with volume unloading). All known attempts to solve this problem have been made precisely under this assumption on the medium [1 to 7]. The system (2.3) is linear and may be solved after having constructed the first approximation.

Solution of the systems (2.2) and (2.3) with the requisite number of arbitrary functions are found by quardatures. For (2.2) we have

$$v^0=c^0(t)\,r^{-2}, \quad \theta_*^0=f(\zeta)=f\left(r^3-3\int c^0(t)\,dt\right), \quad p^0=-\frac{1+2\alpha}{3}\sigma_r^0-\frac{2}{3}\beta$$

$$\theta^0=\theta_*^0, \quad \sigma_\varphi^0=\alpha\sigma_r^0+\beta, \quad \sigma_r^0=r^{-2(1-\alpha)}\left\{\Pi^0(t)+\frac{\beta}{1-\alpha}r^{2(1-\alpha)}+\right.$$

$$\left.+\frac{dc^0(t)}{dt}\int\frac{r^{-2\alpha}\,dr}{1-\theta_*^0(r,t)}-2[c^0(t)]^2\int\frac{r^{-(2\alpha+3)}\,dr}{1-\theta_*^0(r,t)}\right\} \tag{2.4}$$

For the system of second approximations (2.3) we have

$$v^1=r^{-2}\left[\int r^2A(r,t)\,dr+c^1(t)\right], \qquad \theta_*^1=\int B_*(\zeta,t)\,d\zeta+K(t)$$

$$\theta^1=\theta_*^1-\varphi(p^0-p_0,\theta_*^0), \qquad \sigma_r^1=r^{-2(1-\alpha)}\left[\Pi^1(t)+\int D(r,t)\,dr\right]$$

$$\sigma_\varphi^1=\alpha\sigma_r^1, \qquad p^1=-\frac{1+2\alpha}{3}\sigma_r^1 \tag{2.5}$$

$$A(r,t)\equiv\frac{1}{1-\theta_*^0}\left(\frac{\partial}{\partial t}+v^0\frac{\partial}{\partial r}\right)\varphi(p^0-p_0,\theta_*^0), \qquad \zeta\equiv r^3-3\int c^0(t)\,dt$$

$$B_*(\zeta,t)=B(r,t)\equiv-v^1\frac{\partial\theta_*^0}{\partial r}=-3f'(\zeta)\left[\int r^2A(r,t)\,dr+c^1(t)\right]$$

$$D(r,t)\equiv\frac{1}{1-\theta_*^0}\left\{\frac{\partial v^1}{\partial t}+v^0\frac{\partial v^1}{\partial r}+v^1\frac{\partial v^0}{\partial r}+\theta^1\left[\frac{\partial\sigma_r^0}{\partial r}+\frac{2}{r}(\sigma_r^0-\sigma_\varphi^0)\right]\right\}$$

The $c^0(t), c^1(t), \Pi^0(t), \Pi^1(t), f(\zeta), K(t)$ in these formulas are arbitrary functions of their arguments. They should be determined from the boundary conditions of the problem, i.e. from (1.2) to (1.4), (1.7) and (1.16).

We also seek the unknown functions $a(t)$, $R(t)$ in the form of series

$$a=a^0+\varepsilon a^1+\ldots, \quad R=R^0+\varepsilon R^1+\ldots \tag{2.6}$$

The boundary conditions on the surface of the explosion cavity yield:

for the first approximation

$$-\sigma_r^0(a^0,t)=\psi(a^0), \qquad v^0(a^0,t)=(a^0)^{\cdot} \tag{2.7}$$

for the second approximation (2.8)

$$-\sigma_r^1(a,{}^0t)=\left[\psi'(a^0)+\frac{\partial}{\partial r}\sigma_r^0(a^0,t)\right]a^1, \qquad v^1(a^0,t)=\frac{da^1}{dt}-a^1\frac{\partial}{\partial r}v^0(a^0,t)$$

The condition on the shock yield

for the first approximation (2.9)

$$-(\sigma_r^0-\sigma_{re}^0)_{r=R^0}=\rho_0\,[(\theta_*^0-\theta_e^0)\,(R^{0\cdot}-v_e^0)^2]_{r=R^0}$$

$$R^{0\cdot}\,(\theta_*^0-\theta_e^0)_{r=R^0}=[v^0-v_e^0(1-\theta_*^0)]_{r=R^0},\qquad (\sigma_{\varphi e}^0-\alpha\sigma_{re}^0)_{r=R^0}-\beta=0$$

for the second approximation (2.10)

$$-\left[\sigma_r^1-\sigma_{re}^1+R^1\frac{\partial}{\partial r}(\sigma_r^0-\sigma_{re}^0)\right]_{r=R^0}=\rho_0\{2(\theta_*^0-\theta_e^0)(R^{0\cdot}-v_e^0)\,(R^{1\cdot}-v_e^1)+$$

$$+(R^{0\cdot}-v_e^0)^2\,(\theta_*^1-\theta_e^1)+R^1\frac{\partial}{\partial r}\,[(\theta_*^0-\theta_e^0)\,(R^{0\cdot}-v_e^0)^2]\}_{r=R^0}$$

$$[R^{1\cdot}\,(\theta_*^0-\theta_e^0)+R^{0\cdot}\,(\theta_*^1-\theta_e^1)+R^{0\cdot}R^{1\cdot}\frac{\partial}{\partial r}(\theta_*^0-\theta_e^0)]_{r=R^0}=$$

$$=\left\{v^1-v_e^1\,(1-\theta_*^0)+v_e^0\theta_*^1+R^1\frac{\partial}{\partial r}\,[v^0-v_e\,(1-\theta_*^0)]\right\}_{r=R^0}$$

$$\left[\sigma_{\varphi e}^1-\alpha\sigma_{re}^1+R^1\frac{\partial}{\partial r}(\sigma_{\varphi e}^0-\alpha\sigma_{re}^0)\right]_{r=R^0}=0$$

The condition on the shock wave prior to the time of elastic wave origination are obtained from (2.9) and (2.10) if we put $v_e=0$, $\theta_e=0$, $\sigma_{re}=-p_0$.

It is seen from the obtained formulas that the fundamental problem is a nonlinear first-approximation problem. After it has been solved, the construction of the second-approximation solution reduces to the solution of a linear problem.

3. Let us solve the first-approximation problem. Substitution of (2.4) into (2.7), (2.9) yields (3.1)

$$c^0(t)=(a^0)^2\frac{da^0}{dt},\quad \zeta=r^3-(a^0)^3,\qquad \Pi^0(t)=-\left[\psi(a^0)+\frac{\beta}{1-\alpha}\right](a^0)^{2(1-\alpha)}$$

$$p^0(R^0,t)-p_0=-\frac{1+2\alpha}{3}\sigma_r^0(R^0,t)-\frac{2}{3}\beta$$

$$\theta_*^0(R^0,t)=f(\zeta_*)\equiv f\,[(R^0)^3-(a^0)^3]=\theta_*\,[p^0(R^0,t)-p_0]$$

$$-\sigma_r^0(R^0,t)\equiv(R^0)^{-2(1-\alpha)}\left\{\psi(a^0)\,(a^0)^{2(1-\alpha)}+\frac{\beta}{1-\alpha}\left[(a^0)^{2(1-\alpha)}-(R^0)^{2(1-\alpha)}\right]-\right.$$

$$\left.-\frac{1}{3}\frac{d^2(a^0)^3}{dt^2}\int_{a_0}^{R^0}\frac{r^{-2\alpha}\,dr}{1-\theta_*^0(r,t)}+\frac{2}{9}\left[\frac{d(a^0)^3}{dt}\right]^2\int_{a^0}^{R^0}\frac{r^{-(2\alpha+3)}\,dr}{1-\theta_*^0(r,t)}\right\}=$$

$$=\rho_0[\theta_*^0(R^0,t)-\theta_e^0(R^0,t)]\,[R^{0\cdot}-v_e^0(R^0,t)]^2-\sigma_{re}^0(R^0,t)$$

$$R^{0\cdot}\,[\theta_*^0(R^0,t)-\theta_e^0(R^0,t)]=\left(\frac{a^0}{R^0}\right)^2\frac{da^0}{dt}-v_e^0(R^0,t)\,[1-\theta_*^0(R^0,t)]$$

$$\sigma_{\varphi e}^0(R^0,t)-\alpha\sigma_{re}^0(R^0,t)-\beta=0$$

Here $\theta_* = \theta_*(p_*^0 - p_0)$ is a known function describing the loading branch of the diagram for the volume compression of the ground.

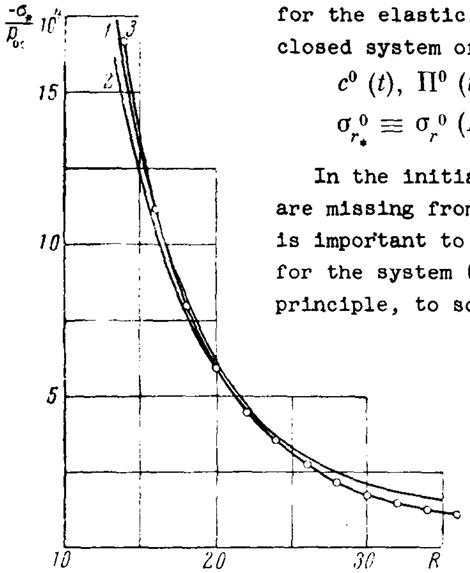

Fig. 2

The relationships (3.1), together with expressions for the elastic quantities (1.13) and (1.14), are a closed system of equations to determine the functions

$$c^0(t),\ \Pi^0(t),\ f(\zeta),\ a^0(t),\ R^0(t),\ \Phi(c_0t - r),$$

$$\sigma_{r_*}^0 \equiv \sigma_r^0(R^0, t),\quad p_*^0 \equiv p^0(R^0, t),\ \theta_*^0(R^0, t)$$

In the initial stage when $R^{0'} \gg c_0$, elastic terms are missing from these equations, i.e. $\Phi \equiv 0$. It is important to stress that the Cauchy problem occurs for the system (3.1), and it is not difficult, in principle, to solve it numerically by using electronic computers.

The procedure for the solution may be simplified if the following is noted. Even for very considerable pressures the quantity θ is very small, i.e. we have $\theta_*^0 \ll 1$ in (3.1), and if θ_*^0 is discarded as compared with unity in the integrands, the integrals will be considtent and the solution to (3.1) simplifies. In particular, even the function θ_*^0 will be determined here. This function may be substituted into the integrals in (3.1) and the process of constructing the solution may be duplicated. Apparently these two iterations will be adequate. From another viewpoint it may be considered that there is a second small parameter $\epsilon_1 = \max \theta_*^0 \ll 1$ in the problem (3.1). Then, by expanding the solution of the problem (3.1) in a power series in ϵ_1 and retaining the first two terms, we obtain the above-mentioned two iterations.

Results of solving the system (3.1) for sandy ground of undisturbed structure are presented in Fig.2 (the necessary experimental data are taken from [12]) for the initial stage when the elastic wave had not yet appeared (*). The nondimensional shock-wave radius (referred to the radius of the charge) is plotted along the horizontal and the nondimensional radial stress on the shock wave along the vertical. Curve 1 corresponds to the exact solution obtained by assigning $\theta_*(R)$ from experiment (in place of the function $\theta_*(p_* - p_0)$; as was remarked by Rykov, assigning $\theta_*(R)$ permits explicit solution of the problem in Lagrangian coordinates in quadratures). Curve 2 corresponds to the first iteration with the approximations obtained from the exact solution of $\theta_*(p_* - p_0)$ by means of

$$\theta_* = [(p_* - p_0) / A]^n$$

(the exact solution and the approximation are given by curves 1 and 2 in Fig.3, respectively). Curve 3 in Fig.2 has been obtained from the same experiments from which the dependence $\theta_* = \theta_*(R)$ was taken for the construction of the exact solution. It is seen that the first iteration yields a good enough approximation to the solution. The existing discrepancy may be associated not so much with the roughness of the first iteration as with the noticeable difference in the dependences $\theta_* = \theta_*(p_* - p_0)$ used in both

*) Computations performed by T.B. Larina on the "Strela" in the Moscow State University Computation Center.

solutions (Fig.3). The discrepancy with experiment for $R > 20$ is apparently connected with neglecting the elastic wave in the computations, since at these distances, the elastic wave is recorded in experiments.

Let us now clarify just how far the obtained method of solution described above, which is based on the assumption that sheer proceeds plastically behind the shock, may be used. Substituting the obtained formulas into the expression for Λ in (1.1), we obtain that the shear will be plastic for the first-approximation solution if the inequality

$$\Lambda^* \equiv \frac{4}{3} G\left[\sigma_r(1-\alpha)-\beta\right]\left(-\frac{3c^0}{r^3}\right)-F'(p)\left(\frac{\partial p}{\partial t}+\frac{c^0}{r^2}\frac{\partial p}{\partial r}\right)>0 \quad (3.2)$$

is satisfied.

Since $\sigma_r < 0, 1-\alpha > 0, \beta > 0$ and as calculations for the initial stage of the motion show, $dp/dt < 0$, then while $a^0 > 0$, i.e. while $da^0/dt > 0$ and $dp/dt < 0$ the shear will be plastic. Only at times close to the time when the explosion cavity ceases to expand (i.e. at the time when $a^0 = 0$) does elastic unloading by means of shear strain begin. In constructing the solution of the problem it is necessary to keep track of the sign of Λ^* and at the time when Λ^* vanishes, construction of elastic unloading waves by means of shear strain should be started from the point where Λ^* vanished. Taking account of the second approximation changes the quantity Λ^* somewhat (but negligibly), hence, such an accounting may affect the value at the mentioned time only inessentially. However, it should be noted that the first term in the expression for Λ^* will be the principal term at this later stage in the motion since it is of the order of $G\sigma_r v/r$ and the second term is of the order of $\sigma_r^2 v/r$, then the ratio between these terms is of the order of $G/\sigma_r \sim G/p_0 \sim 10^3$. Hence, to great accuracy the time of origin of elastic shear unloading agrees with the time of cessation of cavity expansion ($da^0/dt = 0$). Since $\Lambda^* = 0$ at this time for all values of r, the unloading starts simultaneously in all particles from the cavity surface to the shock wave or to particles in which the shock-wave intensity has vanished.

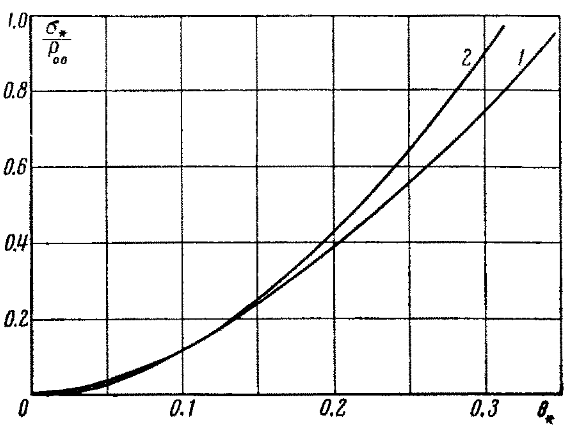

Fig. 3

Starting with this time the motion will be described everywhere by the linear equations of the theory of elasticity, where the modulus of volume compressibility K and the density in the particles which the shock wave has passed will differ from the initial values (K is determined in each particle by the appropriate unloading branch of the volume compression diagram). The difference in the density from the unperturbed value may be neglected because of its smallness ($\theta \ll 1$), while K will change considerably by decreasing from very large values as r increases and passing continuously over into the unperturbed value for $r = R_\infty$. In the domain $r > R_\infty$ the solution of the problem is determined by (1.14) with the as yet unknown function Φ.

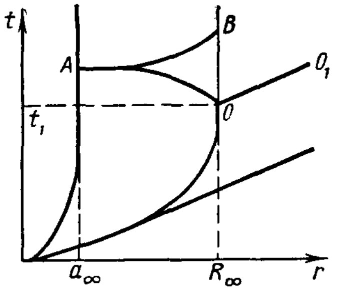

Fig. 4

The motion in the domain $a_\infty \leqslant r \leqslant R_\infty$ will still be described by linear equations, however the coefficients of the equations will be functions of r because of the variability of K, and the equations will be inhomogeneous, hence it is impossible to write down a general solution of type (1.14) for them. The solution may be constructed numerically by the method of characteristics, say. However, a simple approximate solution of the problem may be constructed even for this case. The approximation is that the volume compressibility modulus K, which is very large almost everywhere in the domain $a_\infty \leqslant r \leqslant R_\infty$, is put equal to infinity, which reduces the problem in this domain to a static problem. This solution will be poor directly near the line $t = t_1$ (t_1 is the time in which unloading starts (Fig.4)). Actually, the stress distribution at time t_1, which does not satisfy the equilibrium equations (since only the velocities and not the accelerations are zero at the time $t = t_1$), is changed by a jump by this solution into the distribution satisfying the equilibrium equations (i.e. this solution makes the acceleration zero by a jump). This process occurs gradually in the exact solution, however, because of the slowness of the motion near the cavity the solution obtained here under the assumption of incompressibility of the medium ($K = \infty$) will differ slightly from the exact solution [10] by eliminating a narrow strip near the segments $t = t_1$, $a_\infty \leqslant r \leqslant R_\infty$. In order to estimate the size of the strip it may be considered that its upper bound is a piece of the characteristic reflected from the line $r = a_\infty$ which goes leftward from the point $t = t_1$, $r = R_\infty$ (Fig.4; if $K = \infty$ this strip has zero width, it coincides with the segment $t = t_1$, its upper bound in the exact solution is the curve AB).

The stresses and velocity in the approximate solution will undergo a discontinuity on the characteristic OO_1, and there is no such discontinuity in the exact solution.

The approximate solution may be refined by constructing the line OAB exactly (this is easily done) and by smoothly connecting the values of the stresses at the point O on the segment OB with their values at the point B which have been obtained from the static solution in the domain $a_\infty \leqslant r \leqslant R_\infty$, and which have been kept constant on the line $r = R_\infty$ above the point B. The elastic wave may be constructed by means of the distribution σ_r at $r = R_\infty$ which has been obtained in this manner, i.e. the function Φ may be found for (1.14) in the domain above the characteristic OO_1.

Let us write down the formulas for the described approximate solution. For $K = \infty$ the solution in the domain $a_\infty \leqslant r \leqslant R_\infty^-$ is determined by

$$\sigma_r = 2G\frac{\partial u}{\partial r} - p_0, \qquad \sigma_\varphi = 2G\frac{u}{r} - p_0, \qquad u = \frac{A}{r^2} \tag{3.3}$$

where A = const , u is the additional displacement relative to the state achieved at time $t = t_1$. The boundary condition of the cavity determines the constant

$$A = [\psi(a_\infty) - p_0]\frac{a_\infty{}^3}{4G} \tag{3.4}$$

The stress distribution along the line $r = R_\infty$ (Fig.4) is determined by the function $\sigma_r(R_\infty, t) = f(t)$, which changes smoothly as t changes from t_1 to t_2 , where t_2 is the time corresponding to the point B , from the value of σ_r at the point O , known from the solution of the problem for $t \leqslant t_1$, up to the value at the point B , determined by (3.3), (3.4) for $r = R_\infty$. For $t > t_2$ the quantity σ_r remains constant. A smooth connection may be made by retaining a continuous derivative of σ_r with respect to t at the points O and B (*). By satisfying the condition of continuous σ_r at $r = R_\infty$, we obtain Equation

$$\rho_0 c_0{}^2 \left\{\frac{1}{R_\infty}\Phi''(c_0 t - R_\infty) + 4\gamma^2\left[\frac{1}{R_\infty{}^2}\Phi'(c_0 t - R_\infty) + \right.\right.$$
$$\left.\left. + \frac{1}{R_\infty{}^3}\Phi(c_0 t - R_\infty)\right]\right\} - p_0 = f(t) \tag{3.5}$$

for Φ whose solution is

$$\Phi(\xi) = e^{-\alpha(\xi-\xi_1)}\left[A\sin(\sqrt{\beta-\alpha^2}\,\xi) + B\cos(\sqrt{\beta-\alpha^2}\,\xi)\right] +$$
$$+ \frac{1}{\sqrt{\beta-\alpha^2}}\int_{\xi_1}^{\xi} F(\zeta)\, e^{-\alpha(\xi-\zeta)}\sin\left[\sqrt{\beta-\alpha^2}\,(\xi-\zeta)\right] d\zeta$$
$$F(\zeta) = \frac{R_\infty}{\rho_0 c_0{}^2}\left[f\left(\frac{\zeta + R_\infty}{c_0}\right) + p_0\right], \qquad \xi_1 = c_0 t_1 - R_\infty \tag{3.6}$$
$$\alpha = \frac{2\gamma^2}{R_\infty}, \quad A = \Phi_1 \sin(\sqrt{\beta-\alpha^2}\,\xi_1) + (\Phi_1' + \alpha\Phi_1)\frac{\cos(\sqrt{\beta-\alpha^2}\,\xi_1)}{\sqrt{\beta-\alpha^2}}$$
$$\beta = \frac{4\gamma^2}{R_\infty{}^2}, \qquad B = \Phi_1\cos(\sqrt{\beta-\alpha^2}\,\xi_1) - (\Phi_1' + \alpha\Phi_1)\frac{\sin(\sqrt{\beta-\alpha^2}\,\xi_1)}{\sqrt{\beta-\alpha^2}}$$

The initial values $\Phi_1 = \Phi(\xi_1)$, $\Phi_1' = \Phi'(\xi_1)$ in (3.6) are taken from the solution constructed before for the value $\xi \leqslant \xi_1$.

The complete construction of the solution of the first approximation problem is thereby terminated. It is understood that the exact solution may be constructed for $t > t_1$, the solution for this is constructed numerically by the method of characretistics in the domain $t > t_1$, $a_\infty \leqslant r \leqslant R_\infty$ up to

---

*) A more exact result may be obtained by determining exactly the value of $\partial\sigma_r/\partial t$ at the point O from an analysis of the solution in the neighborhood of the point O and using this value in the junction.

the time t_2 at which $\sigma_r(R_\infty, t)$ practically becomes a constant given by (3.3) and (3.4), and the solution is then continued by using (3.6) with the appropriate initial values $\Phi(\xi_2)$, $\Phi'(\xi_2)$ $(\xi_2 = c_0 t_2 - R_\infty)$.

By having constructed the first-approximation solution, the linear second-approximation problem may be solved by using the quadratures (3.5) deduced above and the boundary conditions (2.8) and (2.10), defining the arbitrary functions.

However, the most interesting properties of the solution, particularly the properties of the emitted elastic wave, are, in practice, determined exactly in the first approximation.

BIBLIOGRAPHY

1. Ishlinskii, A.Iu., Zvolinskii, N.V. and Stepanenko, I.Z., K dinamike gruntovykh mass (On ground mass dynamics). Dokl.Akad.Nauk SSSR, Vol. 94, № 4, 1954.
2. Kompaneets, A.S., Udarnye volny v plasticheskoi uplotniaiushcheisia srede (Shock waves in a plastic compressible medium). Dokl.Akad.Nauk SSSR, Vol.109, № 1, 1956.
3. Rodionov, V.N., Romashov, A.N. and Sukhotin, A.I., Vzryv v uplotniaiushcheisia neogranichennoi srede (Explosion in a compressible unbounded medium). Dokl.Akad.Nauk SSSR, Vol.123, № 4, 1958.
4. Andriankin, E.I. and Koriavov, V.P., Udarnaia volna v peremenno uplotniaemoi plasticheskoi srede (Shock wave in a variably compressive plastic medium). Dokl.Akad.Nauk SSSR, Vol.128, № 2, 1959.
5. Sagomonian, A.Ia., Odnomernye dvizheniia grunta so sfericheskimi, tsilindricheskimi i ploskimi volnami (One-dimensional ground motions with spherical, cylindrical and plane waves). Collection of works "Dinamika gruntov" (Soil Dynamics). Akad.stroit. i arkhit. SSSR, № 4, 1961.
6. Zvolinskii, N.V., Ob izluchenii uprugoi volny pri sfericheskom vzryve v grunte (On the emission (radiation) of an elastic wave from a spherical explosion in the ground). *PMM* Vol.24, № 1, 1960.
7. Lovetskii, E.E., Nekotorye voprosy teorii vzryva v poristom grunte (Some problems of the theory of explosion in porous ground). Izv.Akad.Nauk SSSR, OTN, Mekh. i mashin, № 6, 1959.
8. Grigorian, S.S., Ob obshchikh uravneniiakh dinamiki gruntov (On the general equations of soil dynamics). Dokl.Akad.Nauk SSSR, Vol.124,№2,1959.
9. Grigorian, S.S., Ob osnovnykh predstavleniiakh dinamiki gruntov (On the fundamental concepts of soil dynamics). *PMM* Vol.24, № 6, 1960.
10. Grigorian, S.S., O nekotorykh uproshcheniiakh v opisanii dvizheniia miagkikh gruntov (Some simplifications in the description of the motion of soft soil). *PMM* Vol.27, № 2, 1963.
11. Alekseenko, V.D. Grigorian, S.S., Novgorodov, A.F. and Rykov, G.V., Nekotorye eksperimental'nye issledovaniia po dinamike miagkikh gruntov (Some experimental investigations on soft-ground dynamics). Dokl.Akad. Nauk SSSR, Vol.133, № 6, 1960.
12. Alekseenko, V.D., Grigorian, S.S., Koshelev, L.I., Novgorodov, A.F. and Rykov, G.V., Izmerenie voln napriazhenii v miagkikh gruntakh (Measurement of stress waves in soft soil). PMTF, № 2, 1963.
13. Grigorian, S.S., Liakhov, G.M., Mel'nokov, V.V. and Rykov, G.V., Vzryvnye volny v lessovidnom grunte (Explosion waves in loessy ground). PMTF, № 4, 1963.
14. Alekseenko, V.D., Eksperimental'noe issledovanie dinamicheskogo polia napriazhenii v miagkom grunte pri kontaktnom vzryve (Experimental investigation of the stress field in soft ground during a contact explosion). PMTF, № 5, 1963.

15. Rykov, G.V., Eksperimental'noe issledovanie polia napriazhenii pri vzrýve v peschanom grunte (Experimental investigation of the stress field in an explosion in sandy ground). PMTF, № 1, 1964.

16. Grigorian, S.S. and Chernous'ko, F.L., Odnomernye kvazistaticheskie dvizheniia grunta (One-dimensional quasi-statical motion of soil). *PMM* Vol.25, № 1, 1961.

17. Grigorian, S.S. and Chernous'ko, F.L., Zadacha o porshne dlia uravnenii dinamiki gruntov (The piston problem for the equations of soil dynamics). *PMM* Vol.25, № 5, 1961.

18. Barenblatt, G.I., O rasprostranenii mgnovennykh vozmushchenii v srede s nelineinoi zavisimost'iu napriazhenii ot deformatsii (On the propagation of instantaneous perturbations in a medium with nonlinear stress-strain dependence). *PMM* Vol.17, № 4, 1953.

Translated by M.D.F.